(3o) Αναπλήρωση Δευτέρα 9-11 18/11

Άσκηση 1

1) Έστω τ.δ. $X_1,\dots,X_n \sim B(1,\theta)$

$$f(x_i|\theta) = \theta^{x_i}(1-\theta)^{1-x_i}, \quad x_i = 0,1,\dots$$

Βήτα(α,β) $\quad \pi(\theta) = \dfrac{\theta^{\alpha-1}(1-\theta)^{\beta-1}}{B(\alpha,\beta)}, \quad 0<\theta<1$

$L = T.\Sigma.$

Bayes και να εξετασθεί αν είναι minimax

Λύση

$$\pi(\theta|x) \propto \theta^{\alpha-1}(1-\theta)^{\beta-1}\theta^{\Sigma x_i}(1-\theta)^{n-\Sigma x_i} = \theta^{\alpha+\Sigma x_i-1}(1-\theta)^{n-\Sigma x_i+\beta-1}, \quad 0<\theta<1$$

Άρα $\theta|x \sim$ Βήτα$(\alpha+\Sigma X_i, n-\Sigma X_i+\beta)$

$$d^* = E_{\theta|x}(\theta) \overset{\downarrow}{=} \frac{\alpha+\Sigma X_i}{\alpha+n+\beta}$$

$$R(\theta,d^*) = E_{x|\theta}(d^*-\theta)^2 = Var(d^*) + (E(d^*)-\theta)^2$$

$$= Var\left(\frac{\alpha+\Sigma X_i}{\alpha+n+\beta}\right) + \left(E\left(\frac{\alpha+\Sigma X_i}{\alpha+n+\beta}\right)-\theta\right)^2$$

$$= \frac{n Var(X_i)}{(\alpha+n+\beta)^2} + \left(\frac{\alpha+nE(X_i)}{\alpha+n+\beta}-\theta\right)^2$$

$$= \frac{n\theta(1-\theta)}{(\alpha+n+\beta)^2} + \left(\frac{\alpha+n\theta}{\alpha+n+\beta}-\theta\right)^2 = \frac{n\theta-n\theta^2}{(\alpha+n+\beta)^2} + \frac{\alpha+\cancel{n\theta}-\alpha\theta-\cancel{n\theta}-n\beta}{\alpha+n+\beta}$$

$$= \frac{n\theta-n\theta^2}{(\alpha+n+\beta)^2} + \frac{\alpha^2-2\alpha(\alpha+\beta)\theta+(\alpha+\beta)^2\theta^2}{(\alpha+n+\beta)^2}$$

→ Για να είναι σταθερή συνάρτηση ως προς θ θα πρέπει:

- $n = (\alpha+\beta)^2 \Rightarrow \alpha+\beta = \sqrt{n} \Rightarrow$
- $n = 2\alpha(\alpha+\beta) \Rightarrow n = 2\alpha\sqrt{n} \Rightarrow \alpha = \frac{\sqrt{n}}{2}$ και άρα $\beta = \frac{\sqrt{n}}{2}$

Αν $\theta \sim$ Βήτα$\left(\frac{\sqrt{n}}{2}, \frac{\sqrt{n}}{2}\right)$ τότε minimax εκτιμήτρια

$$d^* = \frac{B(\alpha+\Sigma X_i+1, \beta+n-\Sigma X_i)}{B(\alpha+\Sigma X_i, \beta+n-\Sigma X_i)}$$

Άσκηση 2

Έστω τ.δ. $X_1,\dots,X_n \sim B(1,\theta)$

$$f(x_i|\theta)=\theta^{x_i}(1-\theta)^{1-x_i}, \quad x_i=0,1$$

$$\pi(\theta)=1, \quad 0<\theta<1$$

$$L(\theta,d)=\frac{(d-\theta)^2}{\theta(1-\theta)}$$

Να βρεθεί εκτ. Bayes και πότε είναι minimax;

Λύση

$\pi(\theta|x)\propto \theta^{\Sigma x_i}(1-\theta)^{n-\Sigma x_i}, \quad 0<\theta<1$

$\theta|X \sim$ Βήτα$(\Sigma x_i+1, n-\Sigma x_i+1)$

$\min_d E_{\theta|x}(L(\theta,d))$: Bayes

$\int \frac{(d-\theta)^2}{\theta(1-\theta)}\pi(\theta|x)\,d\theta$ παραγωγ. ως προς d και θέτω $=0$

$$\int \frac{d-\theta}{\theta(1-\theta)}\pi(\theta|x)\,d\theta=0 \Rightarrow d^*=\frac{E_{\theta|x}\left(\frac{1}{1-\theta}\right)}{E_{\theta|x}\left(\frac{1}{\theta(1-\theta)}\right)} \quad (1)$$

$$E_{\theta|x}\left(\frac{1}{1-\theta}\right)=\int_0^1 \frac{\theta^{\Sigma x_i}(1-\theta)^{n-\Sigma x_i-1}}{B(\Sigma x_i+1, n-\Sigma x_i+1)}\,d\theta=$$

$$=\frac{B(\Sigma x_i+1, n-\Sigma x_i)}{B(\Sigma x_i+1, n-\Sigma x_i-1)}\underbrace{\int \frac{\theta^{\Sigma x_i}(1-\theta)^{n-\Sigma x_i-1}}{B(\Sigma x_i+1, n-\Sigma x_i)}\,d\theta}_{1}=$$

$$=\frac{B(\Sigma x_i+1, n-\Sigma x_i)}{B(\Sigma x_i+1, n-\Sigma x_i-1)}$$

άρα
$$d^*=\frac{B(\Sigma x_i+1, n-\Sigma x_i)}{B(\Sigma x_i, n-\Sigma x_i)}=\frac{\frac{\Gamma(\Sigma x_i+1)\Gamma(n-\Sigma x_i)}{\Gamma(\Sigma x_i+1+n-\Sigma x_i)}}{\frac{\Gamma(\Sigma x_i)\Gamma(n-\Sigma x_i)}{\Gamma(\Sigma x_i+n-\Sigma x_i)}}=\frac{\Sigma x_i}{n}=\bar{X}$$

$$R(\theta,d^*)=E_{X|\theta}(L(\theta,d^*))=E_{X|\theta}\left(\frac{(d^*-\theta)^2}{\theta(1-\theta)}\right)=\frac{1}{\theta(1-\theta)}E_{X|\theta}(\bar{X}-\theta)^2$$

$$=\frac{1}{\theta(1-\theta)}\left(\frac{1}{n^2}n\theta(1-\theta)+\cancel{\left(\frac{n\theta}{n}-\theta\right)^2}\right)=\frac{1}{n}$$ δεν

εξαρτάται από το θ άρα είναι

minimax

Άσκηση 3

3) Έστω X τ.μ. από $B(n,\theta)$

$$f(x|\theta)=\binom{n}{x}\theta^x(1-\theta)^{n-x}, \quad x=0,1,\dots,n$$

$$\Pi(\theta)=1 \qquad 0<\theta<1$$

Εκτ. Bayes? Minimax?

Bayes συνάρτ. κινδύνου?

Λύση

$\Pi(\theta|x) \propto \theta^x(1-\theta)^{n-x}$, $0<\theta<1$ άρα $\theta/x \sim Beta(x+1, n-x+1)$

$$d^*=E_{\theta|x}(\theta)=\frac{x+1}{x+1+n-x+1}=\frac{x+1}{n+2}$$

$$R(\theta,d^*)=E_{x|\theta}(L(\theta,d^*))=E_{x|\theta}\left(\frac{x+1}{n+2}-\theta\right)^2$$

$$=Var\left(\frac{x+1}{n+2}\right)+\left(\frac{E(x)+1}{n+2}-\theta\right)^2=\frac{n\theta(1-\theta)}{(n+2)^2}+\left(\frac{n\theta+1}{n+2}-\theta\right)^2$$

$$=\frac{n\theta(1-\theta)}{(n+2)^2}+\frac{(\cancel{n\theta}+1-\cancel{n\theta}-2\theta)^2}{(n+2)^2}=\frac{n\theta-n\theta^2+1-4\theta+4\theta^2}{(n+2)^2}$$

$$=\frac{(4-n)\theta^2+(n-4)\theta+1}{(n+2)^2}$$

Είναι minimax αν-ν $n=4$.

$$R(\pi,d^*)=E_\theta(R(\theta,d^*))=E_\theta\left(\frac{(4-n)\theta^2+(n-4)\theta+1}{(n+2)^2}\right)$$

$$\rightsquigarrow E_\theta(\theta^2)=\int_0^1\theta^2\cancel{\Pi(\theta)}^{1}\,d\theta=\left[\frac{\theta^3}{3}\right]_0^1=\frac{1}{3} \qquad \theta\in(0,1)$$

$$\rightsquigarrow E_\theta(\theta)=\int_0^1\theta\,\cancel{\Pi(\theta)}^{1}\,d\theta=\left[\frac{\theta^2}{2}\right]_0^1=\frac{1}{2}$$

άρα $R(\pi,d^*)=\dfrac{1}{(n+2)^2}\left[(4-n)E_\theta(\theta^2)+(n-4)E_\theta(\theta)+1\right]$

Άσκηση 4

Έστω τ.δ. $X_1, \dots, X_n$:

$X|\theta \sim N(\theta, 1)$

$f(x|\theta) = (2\pi)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(x-\theta)^2\right), \quad x \in \mathbb{R}, \ \theta \in \mathbb{R}$

$\pi(\theta) = (2\pi)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}\theta^2\right), \quad \theta \in \mathbb{R}$

συνάρτ. κινδύνου Bayes για τον εκτιμητή Bayes d^*?

Λύση

Έστω d^* ο εκτ. Bayes

$R(\pi, d^*) = ?$

$\pi(\theta|\tilde{x}) \propto \pi(\theta) \prod_{i=1}^{n} f(x_i|\theta) = \exp\left(-\frac{1}{2}\theta^2\right) \exp\left(-\frac{1}{2}\sum(x_i-\theta)^2\right) =$

$= \exp\left(-\frac{1}{2}\theta^2\right) \exp\left(-\frac{1}{2}\left(\sum x_i^2 - 2\theta \sum x_i + n\theta^2\right)\right)$ άρα

$\pi(\theta|x) \propto \exp\left(-\frac{1}{2}\theta^2\right) \exp\left(-\frac{1}{2}\left(n\theta^2 - 2\theta \sum x_i\right)\right)$

$= \exp\left(-\frac{(n+1)\theta^2}{2} + \frac{1}{2} 2\theta \sum x_i\right)$

$\pi(\theta|x) \propto \exp\left[-\frac{1}{2}\underbrace{(n+1)}_{=\frac{1}{\sigma^2}}\left(\theta^2 - 2\theta\underbrace{\left(\frac{\sum x_i}{n+1}\right)}_{=\mu}\right)\right]$ άρα $\theta|x \sim N\left(\frac{\sum x_i}{n+1}, \frac{1}{n+1}\right)$

άρα $d^* = E_{\theta|x}(\theta) = \frac{\sum x_i}{n+1}$

$R(\pi, d^*) = E_\theta(R(\theta, d^*))$

$R(\theta, d^*) = E_{\theta|x}(d^* - \theta)^2 = \text{Var}\, d^* + (E(d^*) - \theta)^2$

$= \text{Var} \frac{\sum X_i}{n+1} + \left(E\left(\frac{\sum X_i}{n+1}\right) - \theta\right)^2 \qquad X_i \sim N(\theta, 1)$

$= \frac{n}{(n+1)^2} + \left(\frac{n\theta}{n+1} - \theta\right)^2 = \frac{1}{(n+1)^2} + \frac{\theta^2}{(n+1)^2} = \frac{n+\theta^2}{(n+1)^2}$

άρα $R(\pi, d^*) = E_\theta\left(\frac{n+\theta^2}{(n+1)^2}\right) = \frac{n + E(\theta^2)}{(n+1)^2} = \frac{n + \text{Var}(\theta) + E^2(\theta)}{(n+1)^2}$

$\theta \sim N(0,1)$ άρα $R(\pi, d^*) = \frac{1}{n+1}$

Άσκηση 5

$X_1,\dots,X_n$ τ.δ. από την $f(x|\theta)=\frac{2x}{\theta^2}$, $0<x<\theta$

με $\pi(\theta)=1$, $0<\theta<1$

i) $L(\theta,d)=(d-\theta)^2$

ii) $L(\theta,d)=\frac{(d-\theta)^2}{\theta^2}$

Λύση

$$\pi(\theta|x)=\frac{\prod_{i=1}^{n}\frac{2x_i}{\theta^2}}{\int_{X_{(n)}}^{1}\prod\frac{2x_i}{\theta^2}\,d\theta}=\frac{\theta^{-2n}}{\int_{X_{(n)}}^{1}\theta^{-2n}\,d\theta},\quad X_{(n)}<\theta<1$$

$$\left.\begin{array}{l}0<X_1<\theta\\ \vdots\\ 0<X_n<\theta\\ \quad 0<\theta<1\end{array}\right\}\Rightarrow X_{(n)}<\theta<1$$

$$E_{\theta|x}(\theta)=\int_{X_{(n)}}^{1}\theta\,\pi(\theta|x)\,d\theta=\frac{\int_{X_{(n)}}^{1}\theta^{-2n+1}\,d\theta}{\int_{X_{(n)}}^{1}\theta^{-2n}\,d\theta}$$

$\min_d E_{\theta|x}L(\theta,d)$

$\int\frac{(d-\theta)^2}{\theta^2}\pi(\theta|x)\,d\theta$ παραγωγ. και θέτω $=0$

$$\int\frac{d-\theta}{\theta^2}\pi(\theta|x)\,d\theta=0 \Leftrightarrow \pi(\theta|X)=\frac{\prod_{i=1}^{n}\frac{2x_i}{\theta^2}}{\int_{X_{(n)}}^{1}\prod_{i=1}^{n}\frac{2x_i}{\theta^2}}$$

όπου $\pi(\theta|x)=\frac{\theta^{-2n}}{\int_{X_{(n)}}^{1}\theta^{-2n}\,d\theta}$, $X_{(n)}<\theta<1$

$$d^*=\frac{\int_{X_{(n)}}^{1}\frac{1}{\theta}\theta^{-2n}\,d\theta}{\int_{X_{(n)}}^{1}\frac{1}{\theta^2}\theta^{-2n}\,d\theta}$$

Άσκηση 6

Έστω τ.δ. $X_1, \dots, X_n \sim Poisson(\theta)$

$$f(x|\theta) = \frac{\theta^x e^{-\theta}}{x!}, \quad x = 0, 1, \dots$$

$$\pi(\theta) = e^{-\theta}, \quad \theta \geq 0$$

εκτ. Bayes για τις πιθανότητες $P_{X|\theta}(X=k)$

συνάρτ. επιζημίας: το Τ.Σ.

Λύση

$$P_{X|\theta}(X=k) = \frac{\theta^k e^{-\theta}}{k!} = g(\theta)$$

$$\pi(\theta|x) \propto e^{-\theta}\theta^{\Sigma x_i} e^{-n\theta} = \theta^{\Sigma x_i} e^{-(n+1)\theta}$$

$$\Rightarrow \theta|X \sim \Gamma\left(\Sigma x_i + 1, \frac{1}{n+1}\right) \sim$$

$$E_{\theta|x}\left[\frac{\theta^k e^{-\theta}}{k!}\right] = \int_0^{+\infty} \frac{\theta^k e^{-\theta}}{k!} \frac{\theta^{\Sigma x_i} e^{-(n+1)\theta}}{\left(\frac{1}{n+1}\right)^{\Sigma x_i + 1} \Gamma(\Sigma x_i + 1)} d\theta$$

$$= \frac{(n+1)^{\Sigma x_i + 1}\, \Gamma(\Sigma x_i + k + 1)}{\Gamma(\Sigma x_i + 1)\, k!\, (n+2)^{\Sigma x_i + k + 1}} \int_0^{+\infty} \frac{\theta^{\Sigma x_i + k} e^{-(n+2)\theta}}{\left(\frac{1}{n+2}\right)^{\Sigma x_i + k + 1} \Gamma(\Sigma x_i + k + 1)} d\theta$$